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Nonlinear hydrodynamic interface instabilities driven by time-dependent accelerations

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We present a model for nonlinear hydrodynamic instabilities of interfaces and the formation of bubbles driven by time-dependent accelerations $g(t)$. To obtain analytic solutions, we map the equation for the bubble amplitude $\eta(t)$ onto the Schrödinger equation and solve it as an initial value $(\eta_0, \dot{\eta}_0)$ problem in time instead of an eigenvalue problem in space. Very good agreement is obtained with full hydrodynamic simulations. We then apply the WKB approximation to derive scaling with $s=\int\sqrt{g(t)}dt$. Bubbles scale while spikes do not. *Zitterbewegung*, meaning rapid oscillations of $g(t)$ around an average value, has little effect on $\eta(t)$.

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Hydrodynamic instabilities play an important and often critical role in fluids from the microscale ($\sim 10^{-2}$ cm, e.g., movement of biological cells) to the macroscale ($\sim 10^{10}$ cm, e.g., explosion of stars). The instabilities induce small irregularities at fluid interfaces to grow and interpenetrate. Terrestrial experiments ($\sim 10^2$ cm, e.g., rocket-rig experiments [1]) are used to deduce mixing laws. The linear regime, where the amplitude $\eta(t)$ of a perturbation is much smaller than its wavelength λ, is well understood—we expand the Euler (or, more generally, the Navier-Stokes) equations and solve a linear (in η) equation. The nonlinear regime $\eta \gtrsim \lambda$ is more challenging. We can either solve the full hydrodynamic equations numerically or develop models for $\eta(t)$. The latter approach is adopted in this Rapid Communication and compared with the former.

Interfacial instabilities are often driven by a unidirectional acceleration $g(t)$ normal to the interface between two fluids of densities ρ_A and ρ_B. The linear result is

$$\ddot{\eta} - gkA\eta = 0, \tag{1}$$

where $k=2\pi/\lambda$ and the Atwood number $A=(\rho_B-\rho_A)/(\rho_B+\rho_A)$. Although derived for the case $g(t)$=const and known as the Rayleigh-Taylor (RT) instability [2], Eq. (1) applies for arbitrary $g(t)$. Another well-known process is the Richtmyer-Meshkov (RM) instability [3] for the case $g(t)=\Delta v\,\delta(t)$, where Δv is the jump velocity acquired by the interface after the passage of a shock (compressibility effects are ignored in this approach) and $\delta(t)$ is the Dirac delta function. Here we focus on more general time-dependent acceleration histories $g(t)$ because they are used in terrestrial experiments [4]. In addition, inertial confinement fusion capsules ($\sim 10^{-2}$ to 10^{-1} cm in scale) rely on a very carefully designed $g(t)$ necessary to achieve ignition [5,6].

Turning to the nonlinear regime, the most successful model to date has been the Layzer model [7] applied to RT and RM instabilities extended to arbitrary initial amplitudes [8] and Atwood numbers [9]. In Ref. [10], we pointed out several shortcomings of that model.

The resulting equations [Eqs. (8) and (18) in [9] combined as Eq. (6) in [10]) are quite complicated. However, when the initial bubble amplitude η_0 has a special value η^*, the full equations simplify to

$$\ddot{\theta}_L - g(t)k_L A_L \theta_L = 0, \tag{2}$$

where $\theta_L \equiv e^{(\eta-\eta_0)k_L}$, $k_L \equiv c(1+c)(1+A)k/2(1+c+cA-A)$, $A_L \equiv 2A/(1+c+cA-A)$, $\eta^* \equiv 1/k(1+c)$, and $c=2$ for two-dimensional perturbations and $c=1$ for three-dimensional perturbations [10]. Note that dissipation is neglected in Eqs. (1) and (2).

The model we suggest here is to use Eq. (1) for $\eta_0 \le \eta^*$ and switch to Eq. (2) when η reaches η^*. If $\eta_0=\eta^*$ then one uses Eq. (2) only. If $\eta^* < \eta_0 < (\eta_0)_{\max}$, where $(\eta_0)_{\max}=c[1+\sqrt{1+4(1+c)/Ac^2}]/2(1+c)k$, one may revert back to the full equations (they fail for $\eta_0 \ge (\eta_0)_{\max}$; see Ref. [10]). Alternatively, we suggest using Eq. (2) for all $\eta_0 \ge \eta^*$, including $\eta_0 \ge (\eta_0)_{\max}$ for which no other model exists at present. Although such a model is not as accurate as for $\eta_0=\eta^*$, we have found it gives reasonable results within 25–35 % of CALE [11] simulations.

Not only Eq. (2) is simple, it also has the form of Eq. (1) and therefore any solution to the linear η immediately gives the nonlinear solution. Replace k with k_L, A with A_L, and the solution gives θ_L with initial conditions $\theta_L(0)=1$, $\dot{\theta}_L(0)=\dot{\eta}_0 k_L$ (wherever possible, we shall set $\dot{\eta}_0=0$ for brevity). To illustrate, the well-known linear solutions to Eq. (1) are [2,3]

$$\eta(t) = \eta_0 \cosh(\gamma t), \quad \text{RT}, \tag{3}$$

$$\eta(t) = \eta_0 + \dot{\eta}_0 t, \quad \text{RM}, \tag{4}$$

where $\gamma \equiv \sqrt{gkA}$ and $\dot{\eta}_0 = \Delta v kA\eta_0$. One can immediately write down the nonlinear solutions $\eta(t)=\eta_0+(1/k_L)\ln\theta_L$,

$$\eta(t) = \eta_0 + (1/k_L)\ln[\cosh(\gamma_L t)], \quad \text{RT}, \tag{5}$$

$$\eta(t) = \eta_0 + (1/k_L)\ln[1 + \dot{\eta}_0 k_L t], \quad \text{RM}, \tag{6}$$

where $\gamma_L \equiv \sqrt{gk_L A_L}$.

Analytic solutions. We have solved Eqs. (1) and (2) analytically for several acceleration histories $g(t)$ all of which we believe are easily accessible to the linear electric motor (LEM) [4]. Details of the solutions and comparisons with CALE simulations of such *gedanken* experiments will be given elsewhere—only the salient features are discussed here.

We first map Eq. (1) or Eq. (2) onto the Schrödinger equation and solve an initial value instead of an eigenvalue problem.

Start with the harmonic oscillator,

$$g(t) = g_0(1 + \alpha t^2), \tag{7}$$

define $\Lambda=(g_0kA/\alpha)^{1/2}$, $y=(\alpha\Lambda)^{1/2}t$, $\eta=\eta_0 e^{(1/2)y^2}H(y)$, and substitute in Eq. (1) to obtain $d^2H/dy^2+2ydH/dy+(1-\Lambda)H=0$ and recognize $H(y)$ as a Hermite polynomial $H=a_0+a_1y+\ldots+a_ny^n+\ldots$ with

$$a_{n+2} = \frac{\Lambda - 1 - 2n}{(n+2)(n+1)}a_n. \tag{8}$$

Two initial conditions are needed: $H(0)=a_0=1$ and $H'(0)=a_1=\dot{\eta}_0/\eta_0(\alpha\Lambda)^{1/2}$. The remaining a_i are determined by Eq. (8), linking all even-i terms to a_0 and all odd-i terms to a_1.

The usual replacements yield the nonlinear solution $\eta=\eta_0+k_L^{-1}\ln[e^{1/2y^2}H(y)]$ with $\Lambda\to\Lambda_L\equiv(g_0k_LA_L/\alpha)^{1/2}$. Note that if $\dot{\eta}_0=0$ and $\Lambda_L=1$ then *all* $a_i=0$ except $a_0=1$, hence $H(y)=1$ and $\eta=\eta_0+(1/2k_L)y^2=\eta_0+(\alpha/2k_L)t^2$, the solution given in [10].

Next consider

$$g(t) = \dot{g}t. \tag{9}$$

This was used in several LEM experiments with intervals of constant $\dot{g}$. The value of $\dot{g}$ may vary in magnitude and/or in sign from one interval to the next, but within each interval it is constant [4].

CALE simulations of such piecewise linear accelerations give very good agreement with the model. Here we focus on analytic results. Defining $z\equiv(\dot{g}kA)^{1/3}t\equiv t/T$, Eq. (1) reduces to $d^2\eta/dz^2-z\eta=0$ and the solution is written in terms of Airy functions [12]

$$\eta(t) = \alpha\text{Ai}(z) + \beta\text{Bi}(z), \tag{10}$$

with the coefficients α and β determined by the initial conditions η_0 and $\dot{\eta}_0$ as usual.

Let us consider an impulsive acceleration: g increases linearly from $g=0$ to a maximum $g_{\max}=\dot{g}\tau$ by $t=\tau$ then decreases linearly back to 0 by $t=2\tau$. Figure 1 shows an example. The question is: what is $\eta(2\tau)$ and $\dot{\eta}(2\tau)$, the values at the end of the impulse? The answer, in the limit $\tau\to 0$ $g_{\max}\to\infty$ was given by Richtmyer: $\eta_+=\eta_0$ and $\dot{\eta}_+=\Delta vkA\eta_0$ where $\Delta\text{v}=\int gdt=g_{\max}\tau=\dot{g}\tau^2$. We are seeking corrections to his formulae.

During the first interval $0\leq t\leq\tau$, we evolve Eq. (10) using $\eta(0)=\eta_0$ and $\dot{\eta}(0)=0$ and obtain $\eta(\tau)$ at peak acceleration. We then use this $\eta(\tau)$ and $\dot{\eta}(\tau)$ as initial conditions to evolve Eq. (10) over the second interval $\tau\leq t\leq 2\tau$. We find

$$\eta_+ = \eta(2\tau) = \eta_0(1 + \Delta vkA\tau + \ldots), \tag{11}$$

$$\dot{\eta}_+ = \dot{\eta}(2\tau) = \eta_0\Delta vkA(1 + 7\Delta vkA\tau/30 + \ldots). \tag{12}$$

The leading terms agree with Richtmyer's results and the corrections, due to finite τ, are given to order $(\tau/T)^3=\Delta vkA\tau$. Exact results will be given elsewhere.

One can use Eq. (10) to find the asymptotic nonlinear bubble velocity $\dot{\eta}_\infty$ for this case. Since $\eta=\eta_0+(1/k_L)\ln\theta_L$ and $\theta_L=\alpha\text{Ai}+\beta\text{Bi}$, we need only $\text{Bi}(\infty)$ because Ai is a decreasing function [12]. Now,

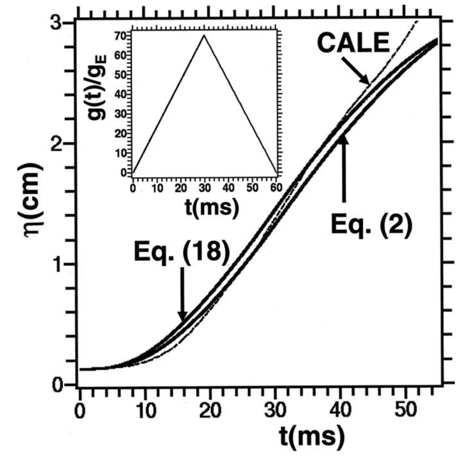

FIG. 1. CALE simulations of a *gedanken* LEM experiment with the acceleration history shown in the inset. $\eta(t)$ is plotted versus time as calculated by CALE, model equation (2), and the WKB approximation (18).

$$\text{Bi}(z) \to \frac{e^{(2/3)z^{3/2}}}{\sqrt{\pi z^{1/2}}}, \tag{13}$$

where $z=(\dot{g}k_LA_L)^{1/3}t$. It follows that $\eta\to\frac{2}{3k_L}z^{3/2}$ and $\dot{\eta}\to\frac{\sqrt{z}\dot{z}}{k_L}$. Therefore

$$\dot{\eta}_\infty = (\dot{g}A_L/k_L)^{1/2}t^{1/2} \sim t^{1/2}. \tag{14}$$

This will turn out to be a special case of a more general result. If $g(t)=g_nt^n$ with a constant g_n then $\dot{\eta}_\infty\sim t^{n/2}$ (see our last equation).

Many other $g(t)$'s are amenable to analytic solutions but will not be presented here. Instead, a more general and much simpler but approximate solution will be given next.

WKB solutions. Define the variable $s(t)$ by

$$s(t) = \int_0^t \sqrt{g(t)}dt, \tag{15}$$

which clearly applies for $g>0$ only. This is not any additional restriction because the Layzer model does not apply to negative, i.e., stable accelerations [10]. In terms of s, Eq. (2) reads as

$$\frac{d^2\theta_L}{ds^2} - k_LA_L\theta_L + \frac{1}{2g^2}\frac{dg}{dt}\frac{d\theta_L}{dt} = 0. \tag{16}$$

In the WKB approximation, we drop the last term in the above equation and obtain

$$\theta_L = \cosh(s\sqrt{k_LA_L}), \tag{17}$$

assuming $\dot{\eta}_0=0$ (otherwise, a sinh term must be added). Therefore,

$$\eta(t) = \eta_0 + (1/k_L)\ln[\cosh(s\sqrt{k_LA_L})] \tag{18}$$

is the nonlinear solution for any $g(t)$ for which the last term in Eq. (16) can be dropped. All information about a particular $g(t)$ is encapsulated in $s(t)$. In other words, if one plots η as a function of s (instead of t) then a universal curve [Eq. (18)] is obtained for all accelerations. We have verified this by subjecting the LEM tank to various acceleration histories—the resulting η's as calculated by CALE fall within 10–15 % of each other when plotted as functions of s.

Such a scaling variable was suggested first by Read who found the turbulent mixing bubble width $h \sim s^2$ [1] and was confirmed more recently by Dimonte and Schneider [4]. These were phenomenological observations based on experiments on turbulent mix. Equation (18), $\eta \sim s\sqrt{A_L/k_L}$, is another scaling expression for the nonlinear single-scale bubble amplitude derived by applying the WKB approximation to Eq. (2).

As found in Ref. [4], scaling does not work for an impulsive acceleration because s remains constant when $g=0$. Fortunately, we have the answer to this RM case: Eq. (6). All we have to do is record the amplitude and its rate of change coming out of any impulse and use them as η_0 and $\dot{\eta}_0$ in Eq. (6) for the subsequent $g=0$ phase.

An important relation follows from Eq. (18):

$$\dot{\eta}(t) = \sqrt{g}\,d\eta/ds = \sqrt{gA_L/k_L}\tanh(s\sqrt{k_L A_L}), \tag{19}$$

which asymptotes to

$$\dot{\eta}_\infty = \sqrt{g(t)A_L/k_L}. \tag{20}$$

Except for case $g=0$ where $\dot{\eta}_\infty = 1/k_L t$ [see Eq. (6)], the above expression agrees with all previously known $\dot{\eta}_\infty$, in particular, for $g=\text{const}$ and $g=\dot{g}t$ where we obtained $\dot{\eta}_\infty$ [Eq. (14)] by analyzing the asymptotic behavior of the second Airy function $\mathrm{Bi}(z)$.

The exact solutions to the "potentials" generally entail hypergeometric functions. The WKB solutions are much simpler. All are given by Eq. (18) with various $s(t)$. For example, when $g(t)=\dot{g}t$ instead of the Airy functions use Eq. (18) with $s=\frac{2}{3}\sqrt{\dot{g}}t^{3/2}$. More generally, $s=\sqrt{g_n}t^{1+n/2}/(1+n/2)$ for $g(t)=g_n t^n$. In special cases, it is possible for the exact solution to be simpler than the WKB. For instance, we derived the exact solution $\eta(t)=\eta_0+(\alpha/2k_L)t^2$ when $g(t)$ is given by Eq. (7) and $\alpha=g_0 k_L A_L$. The WKB scaling variable is a somewhat more complicated expression $s=(\alpha g_0)^{1/2}(t(t^2+1/\alpha)^{1/2}+(1/\alpha)\ln\{\alpha^{1/2}[t+(t^2+1/\alpha)^{1/2}]\})/2$. The advantage of the WKB, however, is that this $s(t)$ is valid for arbitrary α whereas the exact general solution called for Hermite polynomials.

We illustrate with an impulsive acceleration of the LEM tank as shown in Fig. 1: $g(t)$ climbs to $70g_{Earth}$ in 30 ms and returns to 0 at 60 ms. Similar profiles were used for the experiments on turbulent mix [4]. We used $\lambda=7.3/3$ cm, 7.3 cm being the width of the tank, and $\eta_0=0.13$ cm as the initial perturbation between the two fluids (hexane and a water-NaI solution) having $A=0.48$. Figure 1 shows the evolution of $\eta(t)$ calculated by CALE, the exact solution to Eq. (2), and the WKB solution (18). The agreement among all three methods is quite good. We should mention that it is even better for other profiles where the acceleration increases uniformly with time or asymptotes to a constant.

Interestingly, we find that spikes do not scale, and the higher the Atwood number A the more severe is the scale breaking. An analysis (omitted here) reveals the spike *curvature* to be the culprit. The source of the last term in Eq. (16) is multiplied by the curvature denoted by η_2 in [9,10]. While η_2 is constant or asymptotes to a constant for bubbles, it grows ($\eta_2 \sim e^{|\eta k|}$) for spikes and therefore that last term in Eq. (16) cannot be neglected which, clearly, breaks the scaling. Experiments give only a hint that spikes may break scaling at $A\approx 0.22$ [4]. We predict scaling to be severely broken by spikes at high A.

Zitterbewegung. We refer to rapid oscillations of $g(t)$ around an average value as *zitterbewegung*. Experimentally, it is observed that the acceleration of the tank deviates in a small but random manner from the desired or programmed acceleration profile [4]. The deviations are small and, as we shall see, harmless. Our interest was spurred by a quite different and somewhat unexpected observation. When we program the code CALE so that the rigid boundaries at the top and the bottom of the tank undergo a prescribed acceleration such as the one shown in Fig. 1, the interface itself, which is near the middle of the tank, follows the prescribed $g(t)$ with oscillations. These oscillations are generated by sound waves traversing back and forth between the boundaries of the tank, and their amplitude and frequency depend on the equation of state (EOS) used for the two fluids. For simplicity, we used ideal EOSs with "γ"$=100$, a high value needed to ensure incompressibility ("γ" denotes specific-heat ratio, not to be confused with any growth rate). With a higher "γ" such as 500, the oscillations decrease in magnitude and increase in frequency as expected (sound speed $\sim\sqrt{''\gamma''}$), but the computer run times also increase (by about the same factor $\sim\sqrt{5}$). When we changed "γ" and hence the details of the interface acceleration we found, to our surprise, that $\eta(t)$ did not change appreciably. We then solved Eq. (2) using this code-generated numerical $g(t)$, including sound waves and all, and found that the solution was essentially the same as when the idealized acceleration was used in algebraic form. These two experiences (varying "γ" in the code, and solving Eq. (2) with and without oscillations) lead to the conclusion that *zitterbewegung* does not appreciably affect $\eta(t)$; using an average acceleration $\langle g\rangle$ gives an equally good answer.

We illustrate with an extreme example of deliberately imposed large oscillations $g=g_{\max}\sin^2\omega t$, with $\langle g\rangle = g_{\max}/2$. The CALE results with $g_{\max}=70g_{Earth}$ and $\omega=2\pi/14$ ms^{-1} is shown in Fig. 2 and compared with three models: Eq. (2), Eq. (5) with $\langle g\rangle = g_{\max}/2=35g_{Earth}$, and the WKB solution (18) with $s(t)=\sqrt{g_{\max}}\{2(t-r)/\pi+[1-\cos(\omega r)]/\omega\}$, where $r\equiv \mathrm{mod}(t,\pi/\omega)$. Note that Eqs. (2) and (5), indistinguishable in Fig. 2, agree very closely with CALE and all show little oscillations in $\eta(t)$. The WKB solution displays some oscillations but agrees with CALE within 15%. We should point out that this is better than expected because this $g(t)$ periodically goes through 0 and dropping the last term in Eq. (16) is questionable ($\dot{g}/g^2 \sim \cos\omega t/\sin^3\omega t$). The classical RT formula (5) with $\langle g\rangle = g_{\max}/2 = 35g_{Earth}$ is simpler than the WKB formula and shows even closer agreement with CALE. It is striking that oscillations of $\pm 100\%$ around this average value have practically no effect on $\eta(t)$.

Conclusions. Our model for nonlinear hydrodynamic in-

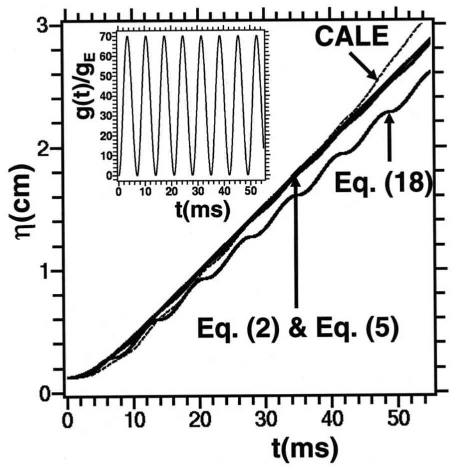

FIG. 2. Same as Fig. 1 for the acceleration shown in the inset. The model [Eq. (2)] uses the full acceleration $g(t)=70g_E\sin^2(\pi t/7)$, t in ms. Equation (5) uses a constant $g=\langle g\rangle=35g_E$. These two curves are hard to distinguish. The WKB result [Eq. (18)] uses $s(t)$ given in the text.

stabilities generated by a time-dependent acceleration [Eq. (2)] has the advantage of being a linear ordinary differential equation (ODE) easily mapped onto the Schrödinger equation with $g(t)$ playing the role of $V(x)$. We discussed a few exact solutions including finite-pulse-width corrections to Richtmyer's formulae for postshock amplitude η_+ and growth rate $\dot{\eta}_+$. We also used a WKB approximation to find an explicit analytic expression for $\eta(t)$: Eq. (18). Even when the required integral [Eq. (15)] cannot be obtained analytically or when $g(t)$ is an accelerometer-measured sequence of points, performing the numerical quadrature in Eq. (15) is simpler than solving the ODE (2). More importantly, it reveals the scaling properties of $\eta(t)$, where it works and where it fails, thus helping us understand the role $s(t)$ plays in experiments on turbulent mix. Finally, we found $\eta(t)$ to be a "robust" quantity not much affected by oscillations of $g(t)$ around an average value. Although spikes do not scale with $s(t)$, we found them to be equally robust against *zitterbewegung*.

We believe that at present there are no other models for nonlinear instabilities driven by a time-dependent $g(t)$. We have used CALE to verify that the model gives reasonable answers. In our simulations, we have chosen values for k and A and patterned the accelerations $g(t)$ after past experiments with the hope that they will be performed and be the ultimate test of the usefulness of this and future models.

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